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# Saturation of the approximation by spectral decompositions

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## 1 Introduction.

Let  $\Omega$  be an open domain in the  $n$  dimensional Euclidean space  $\mathbf{R}^n$ . Consider the operator  $A = -\Delta$  in  $L^2(\Omega)$  with the domain of definition  $D(A) = C_c^\infty(\Omega)$ , where  $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$  is the Laplacian. Denote by  $\hat{A}$  a nonnegative selfadjoint extension of  $A$ . Let  $\{k_\lambda(t)\}$  be a family of bounded piecewise smooth functions on  $[0, \infty)$ . Suppose we have two constants  $\kappa_1, \kappa_2 > 0$  such that  $k_\lambda(t)\sqrt{t}^{n/2-2\kappa_2+1/2} \in L^1(0, \infty)$ ,  $(k_\lambda(t) - 1)/\lambda^{-\kappa_1} t^{\kappa_2}$  are uniformly bounded in  $\lambda$  and  $t \in [0, \infty)$ , and  $(k_\lambda(t) - 1)/\lambda^{-\kappa_1} t^{\kappa_2}$  converge to a nonzero constant as  $\lambda \rightarrow \infty$  for any  $t \in [0, \infty)$ . Let

$$I_\lambda(r) = \int_0^\infty k_\lambda(t^2) J_\nu(rt) t^{\nu+1} dt,$$

where  $\nu = n/2 - 2\kappa_2 + 1$  and  $J_\nu$  is the Bessel function of order  $\nu$ . We assume, furthermore, the following conditions

$$(1.1) \quad \int_0^R s^{2\kappa_2-1} ds \left| \int_s^R r^{n/2-2\kappa_2+2} I_\lambda(r) dr \right| = O(\lambda^{-\kappa_1}),$$

$$(1.2) \quad \left| \int_R^\infty r^{\nu+1} I_\lambda(r) dr \right| = o(\lambda^{-\kappa_1}),$$

and

$$(1.3) \quad \left( \sum_{T=0}^\infty T^{4\kappa_2-3} \max_{T \leq s \leq T+1} \left| \int_R^\infty J_\nu(sr) I_\lambda(r) r dr \right|^2 \right)^{1/2} = o(\lambda^{-\kappa_1})$$

as  $\lambda \rightarrow \infty$  for any small  $R > 0$ .

We shall consider the approximation operator  $k_\lambda(\hat{A})$  for  $f \in L^2(\Omega)$ . We say  $\Delta f \in L^\infty_{\text{loc}}(\Omega)$  if for every compact set  $K$  in  $\Omega$  there is a constant  $C_K$  such that

$$\left| \int_K f(x) \Delta g(x) dx \right| \leq C_K \|g\|_{L^1(K)}$$

for any infinitely differentiable function  $g$  whose support is contained in  $K$ . Let  $\{\varphi_\varepsilon\}$  be an infinitely differentiable approximate identity with supports contained in  $\{x; |x| < \varepsilon\}$ . For a function  $f$  on  $\Omega$  and  $x \in \Omega$ ,  $f$  is said to be regulated at  $x$  if  $f * \varphi_\varepsilon(x) \rightarrow f(x)$  as  $\varepsilon \rightarrow 0^+$ .

In 1970, Igari proved the following Theorem in [5].

**Theorem A.** *Suppose that there exist a complete orthonormal system  $\{u_j\}$  of smooth functions in  $L^2(\Omega)$  and a numerical sequence  $\{\lambda_j\}$  for which  $-\Delta u_j = \lambda_j u_j$  in  $\Omega$ . Let*

$$f_j = \int_\Omega f(x) \overline{u_j(x)} dx, \quad f \in L^2(\Omega)$$

and

$$s_\lambda^\delta f = \sum_{\lambda_j \leq \lambda} \left(1 - \frac{\lambda_j}{\lambda}\right)^\delta f_j u_j, \quad f \in L^2(\Omega).$$

Let  $\delta \geq (n+3)/2$  and  $f \in L^2(\Omega)$  be regulated in  $\Omega$ . Then the following hold.

(i) *The following conditions are equivalent.*

(ia)

$$\|s_\lambda^\delta f - f\|_{L^\infty(K)} = O(\lambda^{-1})$$

as  $\lambda \rightarrow \infty$  for every compact set  $K$  in  $\Omega$ .

(ib)  $\Delta f \in L^\infty_{\text{loc}}(\Omega)$ .

(ii) *The following conditions are equivalent.*

(ii a)

$$\|s_\lambda^\delta f - f\|_{L^\infty(K)} = o(\lambda^{-1})$$

as  $\lambda \rightarrow \infty$  for every compact set  $K$  in  $\Omega$ .

(ii b)  $\Delta f$  vanishes in  $\Omega$ .

Our aim is to give a generalization of Theorem A. Let  $\{k_\lambda(t)\}$  be a family of bounded Borel functions on  $[0, \infty)$ . We can define the bounded operator  $k_\lambda(\hat{A})$  in  $L^2(\Omega)$ .

**Example 1.** Suppose that there exist a complete orthonormal system  $\{u_j\}$  of smooth functions in  $L^2(\Omega)$  and a sequence  $\{\lambda_j\}$  such that  $-\Delta u_j = \lambda_j u_j$  in  $\Omega$ . Let

$$f_j = \int_\Omega f(x) \overline{u_j(x)} dx, \quad f \in L^2(\Omega).$$

Let  $\hat{A}$  be the selfadjoint extension of  $-\Delta$  defined by

$$D(\hat{A}) = \left\{ f \in L^2(\Omega); \sum_{j=1}^{\infty} \lambda_j^2 |f_j|^2 < \infty \right\}$$

and

$$\hat{A}f = \sum_{j=1}^{\infty} \lambda_j f_j u_j, \quad f \in D(\hat{A}).$$

For any  $f \in L^2(\Omega)$  the spectral decomposition of  $f$  is given by

$$E((-\infty, t])f = \sum_{\lambda_j \leq t} f_j u_j$$

and  $k_\lambda(\hat{A})$  is defined by

$$k_\lambda(\hat{A})f = \sum_{j=1}^{\infty} k_\lambda(\lambda_j) f_j u_j, \quad f \in L^2(\Omega).$$

**Example 2.** Let  $\Omega = \mathbf{R}^n$ . Let

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbf{R}^n} f(x) e^{-i\xi \cdot x} dx, \quad f \in L^2(\mathbf{R}^n).$$

In this case, there is a unique nonnegative selfadjoint extension  $\hat{A}$  of  $-\Delta$  defined by

$$D(\hat{A}) = \left\{ f \in L^2(\mathbf{R}^n); |\xi|^2 \hat{f}(\xi) \in L^2(\mathbf{R}^n) \right\}$$

and

$$\hat{A}f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbf{R}^n} |\xi|^2 \hat{f}(\xi) e^{ix \cdot \xi} d\xi, \quad f \in D(\hat{A}).$$

Then the spectral decomposition of  $f \in L^2(\mathbf{R}^n)$  is given by

$$E((-\infty, t]) f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{|\xi|^2 \leq t} \hat{f}(\xi) e^{ix \cdot \xi} d\xi$$

and  $k_\lambda(\hat{A})$  is defined by

$$k_\lambda(\hat{A}) f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbf{R}^n} k_\lambda(|\xi|^2) \hat{f}(\xi) e^{ix \cdot \xi} d\xi, \quad f \in L^2(\mathbf{R}^n).$$

For  $\kappa_2 > 0$  and  $1 < p \leq \infty$ , we say  $(-\Delta)^{\kappa_2} f$  belongs to  $L_{\text{loc}}^p(\Omega)$  if for every bounded open set  $G$  in  $\Omega$  with the closure  $\overline{G}$  contained in  $\Omega$ , there is a constant  $C_G$  such that

$$\left| \int_{\overline{G}} f(x) (-\Delta)^{\kappa_2} g(x) dx \right| \leq C_G \|g\|_{L^{p'}(\overline{G})}$$

for any infinitely differentiable function  $g$  with support contained in  $\overline{G}$ , where  $1/p + 1/p' = 1$ .

Our results are stated as follows.

**Main theorem.** Let  $\Omega$  be an open domain in  $\mathbf{R}^n$  and  $\hat{A}$  be a nonnegative selfadjoint extension of  $-\Delta$  in  $\Omega$ . Let  $\{k_\lambda(t)\}$  be a family of bounded piecewise smooth functions on  $[0, \infty)$  and  $\kappa_1, \kappa_2 > 0$  such that  $k_\lambda(t)\sqrt{t}^{n/2-2\kappa_2+1/2} \in L^1(0, \infty)$ ,  $\lambda^{\kappa_1} t^{-\kappa_2} [k_\lambda(t) - 1]$  are uniformly bounded in  $\lambda$  and  $t \in [0, \infty)$ , and  $\lambda^{\kappa_1} t^{-\kappa_2} [k_\lambda(t) - 1]$  converge to a nonzero constant as  $\lambda \rightarrow \infty$  for any  $t \in [0, \infty)$ .

Suppose that  $\{k_\lambda(t)\}$  satisfies the conditions (1.1), (1.2) and (1.3) as  $\lambda \rightarrow \infty$ . Let  $f$  be a regulated function in  $L^2(\Omega)$ . Furthermore, suppose that  $1 < p \leq \infty$  and  $f \in L^p_{\text{loc}}(\Omega)$ . Then the following hold.

(i) The following two conditions are equivalent.

(ia)

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K)} = O(\lambda^{-\kappa_1})$$

as  $\lambda \rightarrow \infty$  for every compact set  $K$  in  $\Omega$ .

(ib)  $(-\Delta)^{\kappa_2} f \in L^p_{\text{loc}}(\Omega)$ .

(ii) Let  $G \subset \Omega$  be any open set.

(iia) Suppose that  $(-\Delta)^{\kappa_2} f$  vanishes in  $G$ . Then

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K)} = o(\lambda^{-\kappa_1})$$

as  $\lambda \rightarrow \infty$  for any compact set  $K \subset G$ .

(iib) If

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K)} = o(\lambda^{-\kappa_1})$$

as  $\lambda \rightarrow \infty$  for any compact set  $K \subset G$ , then  $(-\Delta)^{\kappa_2} f$  vanishes in  $G$ .

If  $\delta > (n+3)/2$  and  $k_\lambda(t) = (1 - t/\lambda^2)_+^\delta$ , then the conditions (1.1), (1.2) and (1.3) are satisfied. Therefore we have the following:

**Corollary 1.** *Let  $\Omega$  be an open domain in  $\mathbf{R}^n$  and  $\hat{A}$  be a nonnegative selfadjoint extension of  $-\Delta$  in  $\Omega$ . Let  $s_\lambda^\delta = (1 - \hat{A}/\lambda^2)_+^\delta$  and  $\delta > (n+3)/2$ . Let  $f$  be a regulated function in  $L^2(\Omega)$ . Suppose that  $1 < p \leq \infty$  and  $f \in L_{\text{loc}}^p(\Omega)$ . Then the following hold.*

(i) *The following are equivalent.*

(ia)

$$\|s_\lambda^\delta f - f\|_{L^p(K)} = O(\lambda^{-2})$$

*as  $\lambda \rightarrow \infty$  for every compact set  $K$  in  $\Omega$ .*

(ib)  $\Delta f \in L_{\text{loc}}^p(\Omega)$ .

(ii) *Let  $G \subset \Omega$  be any open set.*

(iia) *Suppose that  $\Delta f$  vanishes in  $G$ . Then*

$$\|s_\lambda^\delta f - f\|_{L^p(K)} = o(\lambda^{-2})$$

*as  $\lambda \rightarrow \infty$  for any compact set  $K \subset G$ .*

(iib) *If*

$$\|s_\lambda^\delta f - f\|_{L^p(K)} = o(\lambda^{-2})$$

*as  $\lambda \rightarrow \infty$  for any compact set  $K \subset G$ , then  $\Delta f$  vanishes in  $G$ .*

Our main theorem follows from Theorem 1 in §2 and Theorem 2 in §3. Corollary 1 is proved in §4.

## 2 Saturation of the approximation.

Let  $\Omega$  be an open domain in the  $n$ -dimensional Euclidean space  $\mathbf{R}^n$ . Let

$$(2.1) \quad A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

be a differential operator, where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ ,  $D^\alpha = (-i)^{|\alpha|} (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$  and  $a_\alpha \in C^\infty(\Omega)$ . We consider  $A$  as an operator in  $L^2(\Omega)$  with the domain of definition  $D(A) = C_c^\infty(\Omega)$ . Suppose that  $A$  is formally selfadjoint and semibounded. If  $\hat{A}$  is a selfadjoint extension of  $A$  with the same lower bound  $c$ , then  $\hat{A}$  can be represented in the form of

$$\hat{A} = \int_c^\infty t E(dt).$$

Let  $\{k_\lambda(t)\}$  be a family of bounded Borel functions on  $[c, \infty)$ ,  $\kappa_1, \kappa_2 > 0$  and

$$(2.2) \quad \psi_\lambda(t) := \frac{k_\lambda(t) - 1}{\lambda^{-\kappa_1} t^{\kappa_2}}.$$

Suppose that

- (1)  $\psi_\lambda(t)$  are uniformly bounded in  $\lambda$  and  $t \in [c, \infty)$ , and
- (2)  $\psi_\lambda(t)$  converge to a nonzero constant  $C$  as  $\lambda \rightarrow \infty$  for any  $t \in [c, \infty)$ .

**Lemma 1.** *If  $f \in L^2(\Omega)$  and  $g \in D(\hat{A}^{\kappa_2})$ , then  $\lambda^{\kappa_1} (k_\lambda(\hat{A}) f - f, g) \rightarrow C (f, \hat{A}^{\kappa_2} g)$  as  $\lambda \rightarrow \infty$ .*



**Proof.** By the definition of  $k_\lambda(\hat{A})$ , we have

$$\begin{aligned}
\lambda^{\kappa_1} (k_\lambda(\hat{A}) f - f, g) &= \lambda^{\kappa_1} \int_c^\infty [k_\lambda(t) - 1] (E(dt)f, g) \\
&= \int_c^\infty \frac{k_\lambda(t) - 1}{\lambda^{-\kappa_1}} (f, E(dt)g) \\
&= \int_c^\infty \frac{k_\lambda(t) - 1}{\lambda^{-\kappa_1} t^{\kappa_2}} t^{\kappa_2} (f, E(dt)g) \\
&= \int_c^\infty \psi_\lambda(t) t^{\kappa_2} (f, E(dt)g) \\
&= (f, \overline{\psi_\lambda}(\hat{A}) \hat{A}^{\kappa_2} g) \\
&= \int_c^\infty \psi_\lambda(t) (f, E(dt) \hat{A}^{\kappa_2} g).
\end{aligned}$$

Let  $\rho = (f, E(\cdot) \hat{A}^{\kappa_2} g)$  and  $|\rho|$  be the total variation of  $\rho$ . Then

$$\int_c^\infty |\rho|(dt) \leq \|f\|_{L^2(\Omega)} \|\hat{A}^{\kappa_2} g\|_{L^2(\Omega)} < \infty.$$

Therefore, by Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} \lambda^{\kappa_1} (k_\lambda(\hat{A}) f - f, g) &= \lim_{\lambda \rightarrow \infty} \int_c^\infty \psi_\lambda(t) \rho(dt) \\
&= \int_c^\infty \lim_{\lambda \rightarrow \infty} \psi_\lambda(t) \rho(dt) = C \int_c^\infty \rho(dt) = C (f, \hat{A}^{\kappa_2} g).
\end{aligned}$$

Thus Lemma 1 is proved.

Let  $G$  be an open subset in  $\Omega$  with compact closure  $\overline{G}$  and  $1 < p \leq \infty$ . We say  $A^{\kappa_2} f \in L^p(\overline{G})$  if

$$\|A^{\kappa_2} f\|_{L^p(\overline{G})} := \sup_{\|g\|_{L^{p'}(\overline{G})}=1} \left| \int_\Omega f(x) \overline{\hat{A}^{\kappa_2} g(x)} dx \right| < \infty,$$

where  $1/p + 1/p' = 1$  and  $g$  is an infinitely differentiable function whose support is contained in  $\overline{G}$ .

**Theorem 1.** Let  $\Omega$  be an open domain in  $\mathbf{R}^n$  and  $A$  be a formally selfadjoint semi-bounded differential operator with coefficients in  $C^\infty(\Omega)$  given by (2.1). Suppose that  $\hat{A}$  is a selfadjoint extension of  $A$  with the same lower bound  $c$ . Let  $\{k_\lambda(t)\}$  be a family of bounded Borel functions on  $[c, \infty)$  and  $\kappa_1, \kappa_2 > 0$  such that the sequence  $\{\psi_\lambda(t)\}$  of Borel functions on  $[c, \infty)$  given by (2.2) satisfies (1) and (2). Let  $f \in L^2(\Omega)$ ,  $1 < p \leq \infty$  and  $G$  be any open set in  $\Omega$  with compact closure  $\overline{G}$ . Then the following hold.

(i) If

$$\|k_\lambda(\hat{A})f - f\|_{L^p(\overline{G})} = O(\lambda^{-\kappa_1})$$

as  $\lambda \rightarrow \infty$ , then  $A^{\kappa_2}f \in L^p(\overline{G})$ .

(ii) If

$$\|k_\lambda(\hat{A})f - f\|_{L^p(\overline{G})} = o(\lambda^{-\kappa_1})$$

as  $\lambda \rightarrow \infty$ , then  $A^{\kappa_2}f$  vanishes in  $\overline{G}$ .

**Proof.** Let  $g$  be an infinitely differentiable function and  $\text{supp } g$  be the support of  $g$ .

Suppose that  $\text{supp } g \subset \overline{G}$ . Then by Lemma 1

$$(2.3) \quad \lambda^{\kappa_1} (k_\lambda(\hat{A})f - f, g) \longrightarrow C(f, \hat{A}^{\kappa_2}g) \quad \text{as } \lambda \rightarrow \infty.$$

On the other hand, we have

$$(2.4) \quad |\lambda^{\kappa_1} (k_\lambda(\hat{A})f - f, g)| \leq \lambda^{\kappa_1} \|k_\lambda(\hat{A})f - f\|_{L^p(\overline{G})} \|g\|_{L^{p'}(\overline{G})}.$$

If  $\|k_\lambda(\hat{A})f - f\|_{L^p(\overline{G})} = O(\lambda^{-\kappa_1})$  as  $\lambda \rightarrow \infty$ , then by (2.4) for any  $\lambda$

$$|\lambda^{\kappa_1} (k_\lambda(\hat{A})f - f, g)| \leq C' \|g\|_{L^{p'}(\overline{G})}$$

with some constant  $C' > 0$ . Therefore, by (2.3), we have

$$\left| \int_{\Omega} f(x) \overline{\hat{A}^{\kappa_2} g(x)} dx \right| = |(f, \hat{A}^{\kappa_2} g)| \leq C^{-1} C' \|g\|_{L^{p'}(\overline{G})}$$

for any  $g$ . Thus (i) is proved.

If  $\|k_\lambda(\hat{A})f - f\|_{L^p(\overline{G})} = o(\lambda^{-\kappa_1})$  as  $\lambda \rightarrow \infty$ , then in the same way as in (i), (ii) is proved.

**Examples.** (1) Riesz summation: For  $\kappa > 0$  and  $\delta > 0$ , the Riesz summation is given by the multiplier  $k_\lambda(t) = [1 - (t/\lambda^2)^\kappa]_+^\delta$ . In this case,  $(\lambda^2/t)^\kappa [k_\lambda(t) - 1]$  are uniformly bounded in  $\lambda$  and  $t \in [c, \infty)$  with a constant  $c > 0$  and

$$\lim_{\lambda \rightarrow \infty} \frac{k_\lambda(t) - 1}{(\lambda^{-2}t)^\kappa} = \lim_{s \rightarrow +0} \frac{(1 - s^\kappa)^\delta - 1}{s^\kappa} = - \lim_{s \rightarrow +0} \delta (1 - s)^{\delta-1} = -\delta$$

for any  $t \in [c, \infty)$ . Thus  $\kappa_1 = 2\kappa$ ,  $\kappa_2 = \kappa$  and  $C = -\delta$ , where  $C$  is a constant in (2).

(2) Fejér-Korovkin summation is defined by

$$k_\lambda(t) = \begin{cases} \left(1 - \frac{t}{\lambda^2}\right) \cos \frac{\pi t}{\lambda^2} + \frac{1}{\lambda^2} \cot \frac{\pi}{\lambda^2} \sin \frac{\pi t}{\lambda^2} & t < \lambda^2, \\ 0 & t \geq \lambda^2. \end{cases}$$

In this case,  $(\lambda^2/t)^2 [k_\lambda(t) - 1]$  are uniformly bounded in  $\lambda$  and  $t \in [c, \infty)$  and

$$\lim_{\lambda \rightarrow \infty} \frac{k_\lambda(t) - 1}{(\lambda^{-2}t)^2} = \lim_{s \rightarrow +0} \frac{\cos \pi s - 1}{s^2} = \lim_{s \rightarrow +0} \frac{-\cos^2 \pi s - 1}{s^2 (\cos \pi s + 1)} = - \lim_{s \rightarrow +0} \frac{\sin^2 \pi s}{s^2 (\cos \pi s + 1)} = -\frac{\pi^2}{2}$$

for any  $t \in [c, \infty)$ . Thus  $\kappa_1 = 4$ ,  $\kappa_2 = 2$  and  $C = -\pi^2/2$ .

(3) Rogosinski summation is given by

$$k_\lambda(t) = \begin{cases} \cos \frac{\pi t}{2\lambda^2} & t < \lambda^2, \\ 0 & t \geq \lambda^2. \end{cases}$$

In this case,  $(\lambda^2/t)^2 [k_\lambda(t) - 1]$  are uniformly bounded in  $\lambda$  and  $t \in [c, \infty)$  and

$$\lim_{\lambda \rightarrow \infty} \frac{k_\lambda(t) - 1}{(\lambda^{-2}t)^2} = \lim_{s \rightarrow +0} \frac{\cos \frac{\pi}{2}s - 1}{s^2} = - \lim_{s \rightarrow +0} \frac{\sin^2 \frac{\pi}{2}s}{s^2 \left( \cos \frac{\pi}{2}s + 1 \right)} = - \left( \frac{\pi}{2} \right)^2 \cdot \frac{1}{2} = -\frac{\pi^2}{8}$$

for any  $t \in [c, \infty)$ . Thus  $\kappa_1 = 4$ ,  $\kappa_2 = 2$  and  $C = -\pi^2/8$ .

(4) Jackson summation is given by

$$k_\lambda(t) = \begin{cases} 1 - \frac{3}{2} \left( \frac{t}{\lambda^2} \right)^2 + \frac{3}{4} \left( \frac{t}{\lambda^2} \right)^3 & t < \lambda^2, \\ \frac{1}{4} \left( 2 - \frac{t}{\lambda^2} \right)^3 & \lambda^2 \leq t < 2\lambda^2, \\ 0 & t \geq 2\lambda^2. \end{cases}$$

In this case,  $(\lambda^2/t)^2 [k_\lambda(t) - 1]$  are uniformly bounded in  $\lambda$  and  $t \in [c, \infty)$  and

$$\lim_{\lambda \rightarrow \infty} (\lambda^2/t)^2 [k_\lambda(t) - 1] = -3/2. \text{ Thus } \kappa_1 = 4, \kappa_2 = 2 \text{ and } C = -3/2.$$

(5) Gauss-Weierstrass summation: We consider the multiplier  $k_\lambda^W(t) = \exp(-t/\lambda)$ .

The function of  $t(\lambda/t)[k_\lambda(t) - 1]$  is bounded uniformly in  $\lambda$ , and we have

$$\lim_{\lambda \rightarrow \infty} \frac{k_\lambda(t) - 1}{\lambda^{-1}t} = \lim_{s \rightarrow +0} \frac{e^{-s} - 1}{s} = - \lim_{s \rightarrow +0} e^{-s} = -1.$$

Thus  $\kappa_1 = \kappa_2 = 1$  and  $C = -1$ . Poisson summation is given by the function  $k_\lambda^P(t) = \exp(-\sqrt{t}/\lambda)$ , and we have  $\kappa_1 = 1$  and  $\kappa_2 = 1/2$ .

### 3 Estimates of $k_\lambda(\hat{A})f - f$ .

The aim of this section is to prove the following theorem.

**Theorem 2.** *Let  $\Omega$  be an open domain in  $\mathbf{R}^n$  and  $\hat{A}$  be a nonnegative selfadjoint extension of  $-\Delta$  in  $\Omega$ . Suppose that  $K$  is a compact set in  $\Omega$  and  $K'$  is a closed subset of  $K$  with  $\text{dist}(K', K^c) > 0$ . Let  $\{k_\lambda(t)\}$  be a family of bounded piecewise smooth functions on  $[0, \infty)$  such that  $k_\lambda(t) \sqrt{t}^{n/2-2\kappa_2+1/2} \in L^1(0, \infty)$  with a constant  $\kappa_2 > 0$  and  $k_\lambda(0)=1$  for any  $\lambda$ .*

*Suppose that  $\{k_\lambda(t)\}$  satisfies the conditions (1.1), (1.2) and (1.3) with a constant  $\kappa_1 > 0$  and  $0 < R < \text{dist}(K', K^c) > 0$ . Let  $f$  be a regulated function in  $L^2(\Omega)$ . Suppose that  $1 < p \leq \infty$  and  $f \in L^p(K)$ . Then the following hold.*

(i) *If  $(-\Delta)^{\kappa_2} f \in L^p(K)$ , then*

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K')} = O(\lambda^{-\kappa_1}) \quad \text{as } \lambda \rightarrow \infty.$$

(ii) *If  $(-\Delta)^{\kappa_2} f$  vanishes in  $K$ , then*

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K')} = o(\lambda^{-\kappa_1}) \quad \text{as } \lambda \rightarrow \infty.$$

### 3.1 Generalized eigenfunction system.

In order to prove Theorem 2, we shall use the generalized eigenfunction system corresponding to an ordered representation of  $L^2(\Omega)$  associated with the Laplace operator.

We shall begin with several definitions. We consider  $A = -\Delta$  as an operator in  $L^2(\Omega)$  with the domain of definition  $D(A) = C_c^\infty(\Omega)$ . Let  $\hat{A}$  be a nonnegative selfadjoint extension of  $A$ . Let  $\mathfrak{B}$  be the Borel field on  $\mathbf{R}$  and  $E$  be the unique spectral measure corresponding to  $\hat{A}$ . For  $h \in L^2(\Omega)$ , we define the following closed subspace of  $L^2(\Omega)$ :

$$\begin{aligned} H(h) &:= \left\{ F(\hat{A})h; F \text{ is a Borel function on } \mathbf{R} \text{ and } h \in D(F(\hat{A})) \right\} \\ &= \left\{ F(\hat{A})h; F \in L^2(\mathbf{R}, \mathfrak{B}, (E(\cdot)h, h)) \right\}. \end{aligned}$$

If  $f \in H(h)$ , then we can write uniquely  $f = F(\hat{A})h$ , where  $F \in L^2(\mathbf{R}, \mathfrak{B}, (E(\cdot)h, h))$  and

$$\|f\|_{L^2(\Omega)} = \left( \int_{\mathbf{R}} |F(t)|^2 (E(dt)h, h) \right)^{1/2}.$$

Therefore we can define an isomorphism  $U_h$  from  $H(h)$  onto  $L^2(\mathbf{R}, \mathfrak{B}, (E(\cdot)h, h))$  by  $U_h f := F$ , which preserves inner products.

There exist a sequence of functions  $\{h_j\} \subset L^2(\Omega)$  and a sequence of sets  $\{e_j\} \subset \mathfrak{B}$ , called the set of multiplicity, with the following properties (see [3, XII.3.16] or [4, Chap.14]):

(I)

$$L^2(\Omega) = \bigoplus_j H(h_j).$$

That is,  $H(h_j)$  are mutually orthogonal and span  $L^2(\Omega)$ .

(II)  $\mathbf{R} = e_1 \supseteq e_2 \supseteq \dots$ .

(III)  $(E(e)h_j, h_j) = (E(e \cap e_j)h_1, h_1)$  for any  $e \in \mathfrak{B}$ .

By (I), for  $f \in L^2(\Omega)$  we can write uniquely

$$f = \sum_j F_j(\hat{A}) h_j,$$

where  $F_j \in L^2(\mathbf{R}, \mathfrak{B}, (E(\cdot) h_j, h_j))$  and

$$\left( \sum_j \int_{\mathbf{R}} |F_j(t)|^2 (E(dt) h_j, h_j) \right)^{1/2} = \left( \sum_j \|F_j(\hat{A}) h_j\|_{L^2(\Omega)}^2 \right)^{1/2} = \|f\|_{L^2(\Omega)} < \infty.$$

Therefore we can define an isometry  $U$  from  $L^2(\Omega)$  onto  $\bigoplus_j L^2(\mathbf{R}, \mathfrak{B}, (E(\cdot) h_j, h_j))$ , which is equivalent to say

$$L^2(\Omega) \leftrightarrow \left\{ \{F_j\}; F_j \in L^2(\mathbf{R}, \mathfrak{B}, (E(\cdot) h_j, h_j)) \text{ and } \sum_j \int_{\mathbf{R}} |F_j(t)|^2 (E(dt) h_j, h_j) < \infty \right\},$$

and the correspondence is given by  $Uf := \{F_j\}$ . We denote  $F_j =: (Uf)_j$ .

By (III) we have

$$\bigoplus_j L^2(\mathbf{R}, (E(\cdot) h_j, h_j)) = \bigoplus_j L^2(e_j, (E(\cdot) h_1, h_1)).$$

Let  $\rho := (E(\cdot) h_1, h_1)$ . Then  $U$  is an isomorphism from  $L^2(\Omega)$  onto  $\bigoplus_j L^2(e_j, \rho)$  which preserves inner products, that is, for any  $f, g \in L^2(\Omega)$  it holds that

$$(3.1) \quad (f, g)_{L^2(\Omega)} = \sum_j \int_{e_j} (Uf)_j(t) \overline{(Ug)_j(t)} \rho(dt).$$

$U$  is called an ordered representation of  $L^2(\Omega)$  with respect to  $\hat{A}$ .

With these understood, there exists a sequence of functions  $\{u_j(x, t)\}$  defined on the product space of  $\Omega \times \mathbf{R}$  such that the following conditions are satisfied (see [3, XII.3 and XIV.6] or [4, Chap.15]):

- (i) The functions  $u_j(x, t)$  are  $dx \times d\rho(t)$ -measurable and vanish outside  $\Omega \times e_j$ , where  $dx$  is the Lebesgue measure.

(ii) For any fixed  $t \in \mathbf{R}$ , each  $u_j(x, t)$  belongs  $C^\infty(\Omega)$  and satisfies

$$(3.2) \quad -\Delta u_j(x, t) = t u_j(x, t), \quad x \in \Omega.$$

(iii) For each compact subset  $K$  of  $\Omega$  and each bounded Borel set  $e$  in  $\mathbf{R}$

$$\operatorname{ess\,sup}_{x \in K} \int_e |u_j(x, t)|^2 \rho(dt) < \infty.$$

(iv) For each  $f \in L^2(\Omega)$

$$(3.3) \quad (Uf)_j(t) = \int_{\Omega} f(x) \overline{u_j(x, t)} dx,$$

where the integral exists in the sense of  $L^2(e_j, \rho)$ .

(v) For each  $f \in L^2(\Omega)$  and each  $e \in \mathfrak{B}$

$$(3.4) \quad E(e) f(x) = \sum_j \int_e (Uf)_j(t) u_j(x, t) \rho(dt),$$

where the integral exists and the series converges in the sense of  $L^2(\Omega)$ .

$\{u_j\}$  is called the generalized eigenfunction system of  $\hat{A}$  corresponding to  $U$ . By (v), for  $f \in L^2(\Omega)$  we have

$$(3.5) \quad f(x) = \sum_j \int_{\mathbf{R}} (Uf)_j(t) u_j(x, t) \rho(dt)$$

and

$$(3.6) \quad k_\lambda(\hat{A}) f(x) = \sum_j \int_{\mathbf{R}} k_\lambda(t) (Uf)_j(t) u_j(x, t) \rho(dt).$$



### 3.2 Decomposition of $k_\lambda(\hat{A})f - f$ .

Throughout what follows,  $\Omega$  denotes an open domain in  $\mathbf{R}^n$  and  $\hat{A}$  is a nonnegative selfadjoint extension of  $-\Delta$ . Let  $\mathcal{U}$  denote an ordered representation of  $L^2(\Omega)$  with respect to  $\hat{A}$ ,  $\{u_j\}$  the generalized eigenfunction system and  $\rho$  the measure associated with  $\mathcal{U}$ . We denote the gamma function by  $\Gamma$ , the unit sphere in  $\mathbf{R}^n$  by  $S^{n-1}$ , the Lebesgue measure on the unit sphere  $S^{n-1}$  by  $\sigma$  and the surface area  $2\sqrt{\pi}^n/\Gamma(n/2)$  of  $S^{n-1}$  by  $\omega_n$ . Let  $\kappa_2$  be a constant in (1.1), (1.2) and (1.3), and  $\nu = n/2 - 2\kappa_2 + 1$ .

**Lemma 2.** *Let  $f \in L^2(\Omega)$ ,  $x \in \Omega$  and  $R > 0$ . Then*

$$\begin{aligned} & k_\lambda(\hat{A})f(x) - f(x) \\ &= - \sum_j \int_0^\infty t (Uf)_j(t) u_j(x, t) \rho(dt) \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} s ds \\ &+ \sum_j \int_0^\infty \frac{(Uf)_j(t) u_j(x, t)}{\sqrt{t}^\nu} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t}r) r dr \\ &- f(x) \times \frac{1}{2^\nu \Gamma(\nu+1)} \int_R^\infty I_\lambda(r) r^{\nu+1} dr, \end{aligned}$$

where

$$I_\lambda(r) = \int_0^\infty k_\lambda(s^2) J_\nu(rs) s^{\nu+1} ds.$$

**Proof.** First observe that the function  $k_\lambda(t)$  is piecewise smooth on  $[0, \infty)$  and  $k_\lambda(t) \sqrt{t}^{\nu-1}$  is integrable on  $(0, \infty)$ . By Hankel's integral formula ([2, p.73,(60)]), we have

$$\begin{aligned} k_\lambda(t) &= \frac{1}{\sqrt{t}^\nu} \int_0^\infty J_\nu(\sqrt{t}r) r dr \int_0^\infty k_\lambda(s^2) J_\nu(rs) s^{\nu+1} ds \\ &= \frac{1}{\sqrt{t}^\nu} \int_0^\infty I_\lambda(r) J_\nu(\sqrt{t}r) r dr. \end{aligned}$$

Then, by (3.5), (3.6) and the fact that  $k_\lambda(0) = 1$ , we have

$$\begin{aligned}
& k_\lambda(\hat{A}) f(x) - f(x) \\
&= \sum_j \int_0^\infty \{k_\lambda(t) - k_\lambda(0)\} (Uf)_j(t) u_j(x, t) \rho(dt) \\
&= \sum_j \int_0^\infty (Uf)_j(t) u_j(x, t) \rho(dt) \int_0^\infty \left\{ \frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r) r dr \\
&= \sum_j \int_0^\infty (Uf)_j(t) u_j(x, t) \rho(dt) \int_0^R \left\{ \frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r) r dr \\
&\quad + \sum_j \int_0^\infty (Uf)_j(t) u_j(x, t) \rho(dt) \int_R^\infty \left\{ \frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r) r dr.
\end{aligned}$$

Now apply the formula ([7, p.45])

$$\frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} = -t r^\nu \int_0^r \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} s ds.$$

Note that for the second term, we have

$$\begin{aligned}
& \sum_j \int_0^\infty (Uf)_j(t) u_j(x, t) \rho(dt) \int_R^\infty \left\{ \frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r) r dr \\
&= \sum_j \int_0^\infty \frac{(Uf)_j(t) u_j(x, t)}{\sqrt{t}^\nu} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t}r) r dr \\
&\quad - f(x) \times \frac{1}{2^\nu \Gamma(\nu+1)} \int_R^\infty I_\lambda(r) r^{\nu+1} dr.
\end{aligned}$$

Thus we get Lemma 2.

### 3.3 Proof of Theorem 2.

Let  $f$  be a regulated function in  $L^2(\Omega)$ . Let  $K$  be a compact set in  $\Omega$  and  $K'$  be a closed set in  $K$  with  $\text{dist}(K', K^c) > 0$ . We choose  $0 < R < \text{dist}(K', K^c)$ . Let  $\kappa_1$  and  $\kappa_2$  be constants in (1.1), (1.2) and (1.3). Let  $\nu = n/2 - 2\kappa_2 + 1$  and  $1 < p \leq \infty$ .

Suppose that  $f \in L^p(K)$  and  $(-\Delta)^{\kappa_2} f \in L^p(K)$ . By Lemma 2, we have

$$\begin{aligned}
 (3.7) \quad & \left\| k_\lambda(\hat{A}) f - f \right\|_{L^p(K')} \leq \|f\|_{L^p(K')} \times \frac{1}{2^\nu \Gamma(\nu+1)} \left| \int_R^\infty I_\lambda(r) r^{\nu+1} dr \right| \\
 & + \left\| \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \sum_j \int_0^\infty t (Uf)_j(t) u_j(\cdot, t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \right\|_{L^p(K')} \\
 & + \left\| \sum_j \int_0^\infty \frac{(Uf)_j(t) u_j(\cdot, t)}{\sqrt{t}^\nu} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t}r) r dr \right\|_{L^\infty(K')}.
 \end{aligned}$$

**Lemma 3.** *We have*

$$\begin{aligned}
 & \left\| \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \sum_j \int_0^\infty t (Uf)_j(t) u_j(\cdot, t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \right\|_{L^p(K')} \\
 & \leq C \lambda^{-\kappa_1} \|(-\Delta)^{\kappa_2} f\|_{L^p(K)}.
 \end{aligned}$$

**Proof.** Let  $x \in K'$  and  $0 < s < R$ . Put

$$\begin{aligned}
 g_s(y) &= \frac{1}{s^{\nu+1} |y|^{n/2-1}} \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(|y|r) dr, \\
 g_s^x(y) &= g_s(x-y).
 \end{aligned}$$

If  $|y| > s$ , then  $g_s(y) = 0$  ([7, p.404,(6)]). Therefore  $\text{supp } g_s^x \subset K \subset \Omega$ . Then, by (3.3),

we have

$$\begin{aligned}
 (Ug_s^x)_j(t) &= \int_\Omega g_s^x(y) \overline{u_j(y, t)} dy \\
 &= \int_\Omega g_s(y) \overline{u_j(x-y, t)} dy \\
 &= \frac{1}{s^{\nu+1}} \int \overline{u_j(x-y, t)} dy \frac{1}{|y|^{n/2-1}} \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(|y|r) dr \\
 &= \frac{1}{s^{\nu+1}} \int_0^\infty q^{n/2} dq \int_{S^{n-1}} \overline{u_j(x-qw, t)} \sigma(dw) \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(qr) dr.
 \end{aligned}$$

On the other hand, by (3.2),  $u_j(y, t) \in C^\infty(\Omega)$ , and we have  $-\Delta u_j(y, t) = t u_j(y, t)$  for  $y \in \Omega$ . Therefore, by the mean-value formula, we have

$$\int_{S^{n-1}} u_j(x - q w, t) \sigma(dw) = \sqrt{2\pi}^n \frac{J_{n/2-1}(\sqrt{t}q)}{(\sqrt{t}q)^{n/2-1}} u_j(x, t).$$

Thus, by Hankel's formula, we have

$$\begin{aligned} (Ug_s^x)_j(t) &= \frac{\sqrt{2\pi}^n}{\sqrt{t}^{n/2-1} s^{\nu+1}} \overline{u_j(x, t)} \int_0^\infty J_{n/2-1}(\sqrt{t}q) q dq \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(qr) dr \\ &= \frac{\sqrt{2\pi}^n J_{\nu+1}(\sqrt{t}s)}{\sqrt{t}^{n/2} s^{\nu+1}} \overline{u_j(x, t)}. \end{aligned}$$

We can assume that  $f \in C_c^\infty(\Omega)$  by approximation. Then, by (3.1), we have

$$\begin{aligned} &\sum_j \int_0^\infty t (Uf)_j(t) u_j(x, t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \\ &= \frac{1}{\sqrt{2\pi}^n} \sum_j \int_{e_j} t^{\kappa_2} (Uf)_j(t) \overline{(Ug_s^x)_j(t)} \rho(dt) \\ &= \frac{1}{\sqrt{2\pi}^n} \int_\Omega [(-\Delta)^{\kappa_2} f(y)] g_s^x(y) dy \\ &= \frac{1}{\sqrt{2\pi}^n} \int_\Omega [(-\Delta)^{\kappa_2} f(y)] g_s(x - y) dy. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \sum_j \int_0^\infty t (Uf)_j(t) u_j(x, t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \\ &= \frac{1}{\sqrt{2\pi}^n} \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \int_{|y|<s} [(-\Delta)^{\kappa_2} f(x - y)] g_s(y) dy \\ &= \frac{1}{\sqrt{2\pi}^n} \int_0^R s ds \int_s^R I_\lambda(r) r^{\nu+1} dr \int_{|y|<s} [(-\Delta)^{\kappa_2} f(x - y)] g_s(y) dy. \end{aligned}$$

Applying successively Minkowski's inequality for integral, we have

$$\begin{aligned}
& \left\| \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \sum_j \int_0^\infty t (Uf)_j(t) u_j(\cdot, t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \right\|_{L^p(K')} \\
& \leq \frac{1}{\sqrt{2\pi}^n} \int_0^R s ds \left\| \int_s^R I_\lambda(r) r^{\nu+1} dr \int_{|y|<s} [(-\Delta)^{\kappa_2} f(\cdot - y)] g_s(y) dy \right\|_{L^p(K')} \\
& = \frac{1}{\sqrt{2\pi}^n} \int_0^R s ds \left| \int_s^R I_\lambda(r) r^{\nu+1} dr \right| \left\| \int_{|y|<s} [(-\Delta)^{\kappa_2} f(\cdot - y)] g_s(y) dy \right\|_{L^p(K')} \\
& \leq \frac{1}{\sqrt{2\pi}^n} \int_0^R s ds \left| \int_s^R I_\lambda(r) r^{\nu+1} dr \right| \int_{|y|<s} \|(-\Delta)^{\kappa_2} f(\cdot - y)\|_{L^p(K')} |g_s(y)| dy \\
& \leq \frac{1}{\sqrt{2\pi}^n} \|(-\Delta)^{\kappa_2} f\|_{L^p(K)} \int_0^R s ds \left| \int_s^R I_\lambda(r) r^{\nu+1} dr \right| \int_{|y|<s} |g_s(y)| dy.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{|y|<s} |g_s(y)| dy &= \frac{1}{s^{\nu+1}} \int_{|y|<s} \frac{1}{|y|^{n/2-1}} dy \left| \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(|y|r) dr \right| \\
&= \frac{\omega_n}{s^{\nu+1}} \int_0^s q^{n/2} dq \left| \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(qr) dr \right| \\
&= \frac{\omega_n \Gamma((2\nu + n + 2)/4)}{\Gamma(n/2) \Gamma((2\nu - n + 6)/4) s^{\nu+n/2+1}} \\
&\quad \times \int_0^s \left| {}_2F_1\left((2\nu + n + 2)/4, -(2\nu - n + 2)/4; n/2; q^2/s^2\right) \right| q^{n-1} dq \\
&\leq \frac{C_{\kappa_2}}{s^{\nu-n/2+1}},
\end{aligned}$$

where  ${}_2F_1(\alpha, \beta; \gamma; z)$  is Gauss' hypergeometric function. Therefore the last term is bounded by

$$C_{\kappa_2} \|(-\Delta)^{\kappa_2} f\|_{L^p(K)} \int_0^R s^{2\kappa_2-1} ds \left| \int_s^R I_\lambda(r) r^{\nu+1} dr \right|.$$

By the condition (1.1), we get the bound  $C \lambda^{-\kappa_1} \|(-\Delta)^{\kappa_2} f\|_{L^p(K)}$  for the last term. Thus

Lemma 3 is proved.

We shall use the following lemma ([1, p.655]).

**Lemma 4.** *Under the assumptions above, if  $K$  is a compact set contained in  $\Omega$ , then*

$$\left( \sum_j \int_{T \leq \sqrt{t} \leq T+1} |u_j(x, t)|^2 \rho(dt) \right)^{1/2} \leq C_K (T+1)^{(n-1)/2},$$

where  $C_K$  is a constant independent of  $T \geq 0$  and  $x \in K$ .

**Lemma 5.** *We have*

$$\left\| \sum_j \int_0^\infty \frac{(Uf)_j(t) u_j(\cdot, t)}{\sqrt{t}^\nu} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t} r) r dr \right\|_{L^\infty(K)} = o(\lambda^{-\kappa_1})$$

as  $\lambda \rightarrow \infty$ .

**Proof.** We have, by Schwarz's inequality,

$$\begin{aligned} & \left| \sum_j \int_0^\infty \frac{(Uf)_j(t) u_j(x, t)}{\sqrt{t}^\nu} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t} r) r dr \right| \\ & \leq \left( \sum_j \int_{e_j} |(Uf)_j(t)|^2 \rho(dt) \right)^{1/2} \\ & \quad \times \left( \sum_j \int_0^\infty \frac{|u_j(x, t)|^2}{t^\nu} \rho(dt) \left| \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t} r) r dr \right|^2 \right)^{1/2}. \end{aligned}$$

Now, by (3.1), we have

$$\left( \sum_j \int_{e_j} |(Uf)_j(t)|^2 \rho(dt) \right)^{1/2} = \|f\|_{L^2(\Omega)}.$$

By Lemma 4, there exists a constant  $C_K$  such that

$$\begin{aligned} & \left( \sum_j \int_0^\infty \frac{|u_j(x, t)|^2}{t^\nu} \rho(dt) \left| \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t} r) r dr \right|^2 \right)^{1/2} \\ & \leq C_K \left( \sum_{T=0}^\infty T^{4\kappa_2-3} \max_{T \leq s \leq T+1} \left| \int_R^\infty I_\lambda(r) J_\nu(sr) r dr \right|^2 \right)^{1/2} \end{aligned}$$

uniformly in  $x \in K$ . Therefore, by (1.3), we have

$$\left| \sum_j \int_0^\infty \frac{(Uf)_j(t) u_j(x, t)}{\sqrt{t}^\nu} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t} r) r dr \right| = o(\lambda^{-\kappa_1})$$

uniformly in  $x \in K$  as  $\lambda \rightarrow \infty$ . Thus Lemma 5 is proved.

We remark that  $\left| \int_R^\infty I_\lambda(r) r^{\nu+1} dr \right| = o(\lambda^{-\kappa_1})$  by the assumption (1.2).

By (3.7) together with Lemmas 3 and 5,  $\|k_\lambda(\hat{A})f - f\|_{L^p(K')} = O(\lambda^{-\kappa_1})$  as  $\lambda \rightarrow \infty$ .

If  $(-\Delta)^{\kappa_2} f$  vanishes in  $K$ , then by Lemma 3

$$\left\| \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \sum_j \int_0^\infty t (Uf)_j(t) u_j(\cdot, t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \right\|_{L^p(K')} = 0.$$

Therefore, by (3.7) and Lemma 5, we have  $\|k_\lambda(\hat{A})f - f\|_{L^p(K')} = o(\lambda^{-\kappa_1})$  as  $\lambda \rightarrow \infty$ .

Consequently, Theorem 2 is proved.

## 4 Applications of main theorem.

### 4.1 Proof of Corollary 1.

Let  $k_\lambda(t) = (1 - t/\lambda^2)_+^\delta$ . Then we have the formula (see [2, p.92,(34)])

$$k_\lambda(t) = \frac{2^\delta \Gamma(\delta + 1)}{\lambda^{\delta - n/2} \sqrt{t}^{n/2 - 1}} \int_0^\infty \frac{J_{n/2 + \delta}(\lambda r) J_{n/2 - 1}(\sqrt{t} r)}{r^\delta} dr,$$

and can take  $\kappa_2 = 1$ . We have

$$I_\lambda(r) = \int_0^\infty k_\lambda(t^2) J_{n/2 - 1}(rt) t^{n/2} dt = 2^\delta \Gamma(\delta + 1) \lambda^{n/2 - \delta} J_{n/2 + \delta}(\lambda r) r^{-\delta - 1}.$$

To check the conditions (1.1), (1.2) and (1.3), let  $R > 0$  and  $\delta > (n - 3)/2$ . Then we have

$$\left| \int_R^\infty I_\lambda(r) r^{n/2} dr \right| = 2^\delta \Gamma(\delta + 1) \lambda^{n/2 - \delta} \left| \int_R^\infty \frac{J_{n/2 + \delta}(\lambda r)}{r^{\delta - n/2 + 1}} dr \right| \leq C_{\delta, R} \lambda^{(n-3)/2 - \delta}.$$

On the other hand, we have

$$\begin{aligned} \int_0^R s ds \left| \int_s^R I_\lambda(r) r^{n/2} dr \right| &= 2^\delta \Gamma(\delta + 1) \lambda^{n/2 - \delta} \int_0^R s ds \left| \int_s^R \frac{J_{n/2 + \delta}(\lambda r)}{r^{\delta - n/2 + 1}} dr \right| \\ &\leq \begin{cases} C_\delta \lambda^{(n-3)/2 - \delta} & \text{if } (n - 3)/2 < \delta < (n + 1)/2, \\ C_\delta \lambda^{(n-3)/2 - \delta} \log \lambda & \text{if } \delta = (n + 1)/2, \\ C_\delta \lambda^{-2} & \text{if } \delta > (n + 1)/2. \end{cases} \end{aligned}$$

We now apply the estimates (see [6, p.202, Lemma 18.10 a])

$$\begin{aligned} &\left| \int_R^\infty \frac{J_{n/2 + \delta}(\lambda r) J_{n/2 - 1}(s r)}{r^\delta} dr \right| \\ &\leq \begin{cases} C_{\delta, R} \lambda^{-1/2} s^{-1/2} & \text{if } s, \lambda > 0, \\ C_{\delta, R} \frac{\lambda^{-3/2} s^{1/2}}{\lambda - s} + C_{\delta, R} \lambda^{-3/2} s^{-1/2} & \text{if } 0 < s < \lambda, \\ C_{\delta, R} \frac{\lambda^{1/2} s^{-3/2}}{s - \lambda} + C_{\delta, R} \lambda^{-1/2} s^{-3/2} & \text{if } 0 < \lambda < s. \end{cases} \end{aligned}$$



Then we have

$$\begin{aligned}
& \left( \sum_{T=0}^{\infty} T \max_{T \leq s \leq T+1} \left| \int_R^{\infty} I_{\lambda}(r) J_{n/2-1}(sr) r dr \right|^2 \right)^{1/2} \\
&= 2^{\delta} \Gamma(\delta + 1) \lambda^{n/2-\delta} \left( \sum_{T=0}^{\infty} T \max_{T \leq s \leq T+1} \left| \int_R^{\infty} \frac{J_{n/2+\delta}(\lambda r) J_{n/2-1}(sr)}{r^{\delta}} dr \right|^2 \right)^{1/2} \\
&\leq C_{\delta,R} \lambda^{(n-1)/2-\delta}.
\end{aligned}$$

If  $\delta > (n+3)/2$ , then the last term is  $o(\lambda^{-2})$ . Thus Corollary 1 follows from Main theorem.

## 4.2 The Gauss-Weierstrass summation.

Let  $k_{\lambda}^W(t) = e^{-t/\lambda}$  ( $\lambda \rightarrow \infty$ ). We then have

$$(4.1) \quad \int_0^{\infty} k_{\lambda}^W(t^2) J_{\nu}(rt) t^{\nu+1} dt = \int_0^{\infty} e^{-t^2/\lambda} J_{\nu}(rt) t^{\nu+1} dt = \frac{\lambda^{\nu+1} r^{\nu}}{2^{\nu+1}} \exp\left(-\frac{\lambda r^2}{4}\right)$$

(cf. [2, 7.7.3]). Let  $\Omega$  be an open domain in  $\mathbf{R}^n$  and  $\hat{A}$  be a nonnegative selfadjoint extension of  $-\Delta$  in  $\Omega$ .

**Corollary 2.** *Let  $f$  be a regulated function in  $L^2(\Omega)$ . Suppose that  $1 < p \leq \infty$  and  $f \in L_{loc}^p(\Omega)$ . Then the following hold.*

(i) *The following are equivalent.*

(ia)

$$\|k_{\lambda}^W(\hat{A})f - f\|_{L^p(K)} = O(\lambda^{-1})$$

*as  $\lambda \rightarrow \infty$  for every compact set  $K$  in  $\Omega$ .*

(ib)  $\Delta f \in L_{loc}^p(\Omega)$ .

(ii) Let  $G \subset \Omega$  be any open set.

(ii a) Suppose that  $\Delta f$  vanishes in  $G$ . Then

$$\left\| k_{\lambda}^W(\hat{A}) f - f \right\|_{L^p(K)} = o(\lambda^{-1})$$

as  $\lambda \rightarrow \infty$  for any compact set  $K \subset G$ .

(ii b) If

$$\left\| k_{\lambda}^W(\hat{A}) f - f \right\|_{L^p(K)} = o(\lambda^{-1})$$

as  $\lambda \rightarrow \infty$  for any compact set  $K \subset G$ , then  $\Delta f$  vanishes in  $G$ .

**Proof.** For the Gauss-Weierstrass summation method we take  $\kappa_2 = 1$ . Let  $R$  be a small positive number. By (4.1), we have

$$\int_R^{\infty} r^{n/2} dr \int_0^{\infty} k_{\lambda}^W(t^2) J_{\nu}(rt) t^{\nu+1} dt = \left(\frac{\lambda}{2}\right)^{n/2} \int_R^{\infty} r^{n-1} \exp\left(-\frac{\lambda r^2}{4}\right) dr = o(\lambda^{-1}),$$

$$\begin{aligned} & \int_0^R s ds \left| \int_s^R r^{n/2} dr \int_0^{\infty} k_{\lambda}^W(t^2) J_{\nu}(rt) t^{\nu+1} dt \right| \\ &= \left(\frac{\lambda}{2}\right)^{n/2} \int_0^R s ds \int_s^R r^{n-1} \exp\left(-\frac{\lambda r^2}{4}\right) dr = O(\lambda^{-1}) \end{aligned}$$

and

$$\begin{aligned} & \left( \sum_{T=0}^{\infty} T \max_{T \leq s \leq T+1} \left| \int_R^{\infty} J_{n/2-1}(sr) r dr \int_0^{\infty} k_{\lambda}^W(t^2) J_{\nu}(rt) t^{\nu+1} dt \right|^2 \right)^{1/2} \\ &= \left(\frac{\lambda}{2}\right)^{n/2} \left( \int_R^{\infty} r^{n-1} \exp\left(-\frac{\lambda r^2}{2}\right) dr \right)^{1/2} = o(\lambda^{-1}). \end{aligned}$$

Thus Corollary 2 follows from Main theorem.

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